Infinite Series
(Part-2)

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Alternating Series

We now discuss convergence of series whose terms are alternately positive and negative.

Definition 1.

A series which the terms are alternately positive and negative is an alternating series.

Example 2.

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$
2. $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots$
3. $1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots$

Alternating Series

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n+1} u_n \quad \text{or} \quad a_n = (-1)^n u_n$$

where $u_n = |a_n|$ is a positive number.

Series (1), called the **alternating harmonic series**, converges, as we will see in a moment. Series (2), a geometric series with ratio $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series (3) diverges because the nth term does not approach zero.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test. This test is for *convergence* of an alternating series and cannot be used to conclude that such a series diverges. If we multiply $(u_1 - u_2 + u_3 - u_4 + \cdots)$ by -1, we see that the test is also valid for the alternating series $-u_1 + u_2 - u_3 + u_4 - \cdots$, as with the one in Series (2) given above.

Theorem 3 (The Alternating Series Test (Leibniz's Theorem)).

The Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. *The u_n's are all positive.*
2. $u_n \geq u_{n+1}$ *for all* $n \leq N$, *for some integer* N.
3. $u_n \to 0$.

Proof of Leibniz's Theorem

If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero.

Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is non-decreasing and bounded from above, it has a limit, say

$$\lim_{m \to \infty} s_{2m} = L.$$

Proof of Leibniz's Theorem (contd...)

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m\to\infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L.$$

We have the following result for sequences :

For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. If $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

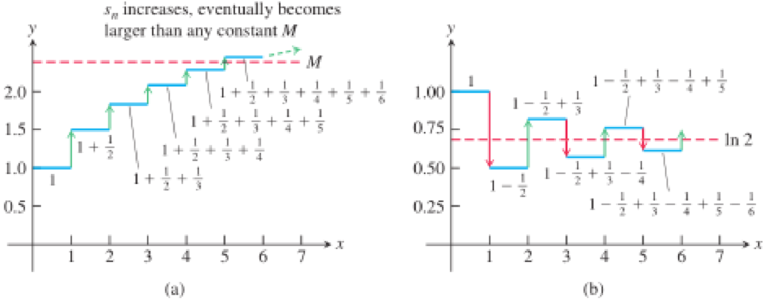
As $s_{2m+1} \to L$ and $s_{2m} \to L$, $\lim_{n\to\infty} s_n = L$.

Example 4 (The alternating harmonic series).

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies the three requirements of Leibniz's Theorem with $N = 1$; it therefore converges by the Alternating Series Test. Notice that the test gives no information about what the sum of the series might be.
The following figure shows histograms of the partial sums of the divergent harmonic series and those of the convergent alternating harmonic series.
It turns out that the alternating harmonic series converges to $\ln 2$. Rather than directly verifying the definition $u_n \geq u_{n+1}$, a second way to show that the sequence $\{u_n\}$ is nonincreasing is to define a differentiable function $f(x)$ satisfying $f(n) = u_n$. That is, the values of f match the values of the sequence at every positive integer n. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is nonincreasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

Graphical Interpretation



(a) The harmonic series diverges, with partial sums that eventually exceed any constant.
(b) The alternating harmonic series converges to $\ln 2 \approx .693$.

An Example of Nonincreasing Sequence

Example 5.

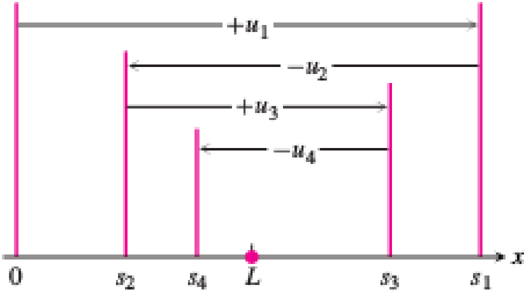
We show that the sequence $u_n = 10n/(n^2+16)$ *is eventually nonincreasing. Define* $f(x) = 10x/(x^2+16)$.

Then from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16-x^2)}{(x^2+16)^2} \leq 0 \quad \textit{whenever} \quad x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ *for* $n \geq 4$. *That is, the sequence* $\{u_n\}$ *is nonincreasing for* $n \geq 4$.

A graphical interpretation of the partial sums shows how an alternating series converges to its limit L when the three conditions are satisfied with $N = 1$. Starting from the origin of the x-axis, we lay off the positive distance $s_1 = u_1$.



Graphical Interpretation

To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2. Since $u_2 \leq u_1$, we do not back up any farther than the origin.

We continue in this seesaw fashion, backing up or going forward as the signs in the series demand. But for $n \geq N$, each forward or backward step is shorter than (or at most the same size as) the preceding step, because $u_{n+1} \leq u_n$. And since the nth term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller.

Graphical Interpretation

We oscillate across the limit L, and the amplitude of oscillation approaches zero. The limit L lies between any two successive sums s_n and s_{n+1} and hence differs from s_n by an amount less than u_{n+1}.

Because $|L - s_n| < u_{n+1}$ for $n \geq N$,

We can make useful estimates of the sums of convergent alternating series.

The Alternating Series Estimation Theorem

Theorem 6 (The Alternating Series Estimation Theorem).

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

satisfies the three conditions of Leibniz's Theorem, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1} u_n.$$

approximates the sum L *of the series with an error whose absolute value is less than* u_{n+1}, *the numerical value of the first unused term. Furthermore, the remainder,* $L - s_n$, *has the same sign as the first as the first unused term.*

Example

Example 7.

Consider the following series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots .$$

The Alternating Series Estimation Theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. *The sum of the first eight terms is* 0.6640625. *The sum of the series is*

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640625 = 0.0026041666\cdots$ *is positive and less than* $(1/256) = 0.00390625$.

The alternating harmonic series does not converge absolutely. The corresponding series of absolute value is the (divergent) harmonic series.

Conditionally Convergent

If we replace all the negative terms in the alternating series in

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n},$$

changing them to positive terms instead, we obtain the geometric series $\sum 1/2^n$. The original series and the new series of absolute values both converge (although to different sums).

For an absolutely convergent series, changing infinitely many of the negative terms in the series to positive values does not change its property of still being a convergent series. Other convergent series may behave differently. The convergent alternating harmonic series has infinitely many negative terms, but if we change its negative terms to positive values, the resulting series is the divergent harmonic series. So the presence of infinitely many negative terms is essential to the convergence of the alternating harmonic series. The following terminology distinguishes these two types of convergent series.

Definition 8 (Conditionally Convergent).

A series that converges but does not converge absolutely is called a conditionally convergent series.

The alternating harmonic series is conditionally convergent, or converges conditionally. The next example extends that result to the alternating p-series.

Alternating p-Series

Example 9.

If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely. If $0 < p \leq 1$, the series converges conditionally.

Conditional convergence: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$.
Absolute convergence: $1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \ldots$.

Summary of Facts about Alternating Series

If $\sum a_n$ is an alternating series, then exactly one of the following holds:

- $\sum a_n$ is convergent (can be concluded by absolute convergence test, in which case both $\sum |a_n|$ and $\sum a_n$ are convergent).
- $\sum a_n$ is conditionally convergent (i.e., $\sum a_n$ is convergent, while $\sum |a_n|$ is not).
- $\sum a_n$ is divergent (in which case, both $\sum a_n$ and $\sum |a_n|$ are divergent. n^{th}-term Test may be helpful most of the times.)

Rearranging Series

We need to be careful when using a conditionally convergent series. We have seen with the alternating harmonic series that altering the signs of infinitely many terms of a conditionally convergent series can change its convergence status. Even more, simply changing the order of occurrence of infinitely many of its terms can also have a significant effect, as we now discuss.

We can always rearrange the terms of a *finite* collection of numbers without changing their sum. The same result is true for an infinite series that is absolutely convergent.

Absolutely convergence is important for two reasons. First, we have good tests for convergence of series of positive terms. Second, if a series converges absolutely, then it converges. That is the thrust of the next theorem.

Theorem 10 (The Rearrangement Theorem for Absolutely Convergent Series).

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Rearranging Series

On the other hand, if we rearrange the terms of a conditionally convergent series, we can get different results. In fact, for any real number r, a given conditionally convergent series can be rearranged so its sum is equal to r.

Here is an example of summing the terms of a conditionally convergent series with different orderings, with each ordering giving a different value for the sum.

Example 11.

We know that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ *converges to some number* L*. Moreover, by Theorem 10,* L *lies between the successive partial sums* $s_2 = 1/2$ *and* $s_3 = 5/6$*, so* $L \neq 0$*. If we multiply the series by 2 we obtain*

$$\begin{aligned} 2L = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right) \\ &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots . \end{aligned}$$

Now we change the order of this last sum by grouping each pair of terms with the same odd denominator, but leaving the negative terms with the even denominators as they are placed (so the denominators are the positive integers in their natural order). This rearrangement gives

$$\begin{aligned} &(2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots \\ &\quad = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right) \\ &\quad = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = L. \end{aligned}$$

Rearranging Series

So by rearranging the terms of the conditionally convergent series

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n},$$

the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is the alternating harmonic series itself. If the two series are the same, it would imply that $2L = L$, which is clearly false since $L \neq 0$.

What is wrong here?

Exercise 12.

Multiplying both sides of the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{2 = 11} - \frac{1}{12} + \dots$$

by 2 to get

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \dots.$$

Collect terms with the same denominator, as the arrows indicate, to arrive at

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots.$$

The series on the right-hand side of this equation is the series we started with. Therefore, $2S = S$*. and dividing by* S *gives* $2 = 1$*. (Source: "Riemann's Rearrangement Theorem "by Stewart Galanor, Mathematics Teacher,Vol.80,No.8,1987, pp. 675-681.)*

Rearranging Series

Example 11 shows that we cannot rearrange the terms of a conditionally convergent series and expect the new series to be the same as the original one.

When we use a conditionally convergent series, the terms must be added together in the order they are given to obtain a correct result.

In contrast, Theorem 10 guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.

Outline of the proof of the The Rearrangement Theorem for Absolutely Convergent Series

Part (a)

Let ε be a positive real number, let $L = \sum_{n=1}^{\infty} a_n$, and let $s_k = \sum_{n=1}^{k} a_n$. Show that for some index N_1 and for some index $N_2 \geq N_1$,

$\sum_{n=N_1}^{\infty} |a_n| < \frac{\varepsilon}{2}$ and $|s_{N_2} - L| < \frac{\varepsilon}{2}$.

Since all the terms $a_1, a_2, \ldots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is index $N_3 \geq N_2$ such that if $n \geq N_3$, then $(\sum_{k=1}^{n} b_k) - s_{N_2}$ is at most a sum of terms a_m with $m \geq N_1$. Therefore if $n \geq N_3$,

$$|\sum_{k=1}^{n} b_k - L| \leq |\sum_{n=1}^{n} b_k - s_{N_2}| + |s_{N_2} - L|$$

$$\leq \sum_{k=N_1}^{\infty} |a_k| + |s_{N_2} - L| < \varepsilon.$$

Outline of the proof of the The Rearrangement Theorem for Absolutely Convergent Series (contd...)

Part (b)

The argument in part (a) shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Now show that because $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ converges to $\sum_{n=1}^{\infty} a_n$.

Example 13 (Applying the Rearrangement Thoerem).

The series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + (-1)^{n-1}\frac{1}{n^2} + \cdots$$

converges absolutely.

A possible rearrangement of the terms of the series might start with a positive term, then two negative terms, then three positive terms, then four negative terms, and so on: After k terms of one sign, take $k+1$ terms of the opposite sign. The first ten terms of such a series look like this:

$$1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \frac{1}{36} - \frac{1}{64} - \frac{1}{100} - \frac{1}{144} + \cdots .$$

Applying the Rearrangement Thoerem

The Rearrangement Theorem says that both series converge to the same value.

In this example, if we had the second series to begin with, we would probably be glad to exchange it for the first, if we knew that we could.

We can do even better: The sum of either series is also equal to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

Applying the Rearrangement Thoerem

If we rearrange infinitely many terms of conditionally convergent series, we can get result that are far different from the sum of the original series. Here is an example.

Example 14 (Rearranging the Alternating Harmonic Seires).

The alternating harmonic seires

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\cdots$$

can be rearranged to diverge to reach any preassigned sum.

Rearranging the Alternating Harmonic Seires

Rearranging $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ to diverge. The series of terms $\sum \frac{1}{2n-1}$ diverges to $+\infty$ and the series of terms $\sum \frac{-1}{2n}$ diverges to $-\infty$. No matter how far out in the sequence of odd-numbered terms we begin, we can always add enough positive terms to get an arbitrarily large sum. Similarly, with the negative terms, no matter how far out we start, we can add enough consecutive even-numbered terms to get a negative sum of arbitrarily large absolute value. If we wished to do so, we could start adding odd-numbered terms until we had a sum greater than $+3$, say, and then follow that with enough consecutive negative terms to make the total less than -4. We could then add enough positive terms to make the total greater than $+5$ and follow with consecutive unused negative terms to make a new total less than -6, and so on. In this way, we could make the swings arbitrarily large in either direction.

Rearranging the Alternating Harmonic Seires

Rearranging $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ to converge to 1. Another possibility is to focus on a particular limit. Suppose we try to get sums that converge to 1. We start with the first term, 1/1, and then subtract 1/2. Next we add 1/3 and 1/5, which brings the total back to 1 or above. Then we add consecutive negative terms until the total is less than 1. We continue in this manner: When the sum is less than 1, add positive terms until the total is 1 or more: then subtract (add negative) terms until the total is again less than 1.

Rearranging the Alternating Harmonic Seires

This process can be continued indefinitely. Because both the odd-numbered terms and the even-numbered terms of the original series approach zero as $n \to \infty$.

The amount by which our partial sums exceed 1 or fall below it approaches zero. So the new series converges to 1. The rearranged series starts like this:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} + \frac{1}{21} - \frac{1}{12} + \frac{1}{23} + \frac{1}{25} - \frac{1}{14} + \frac{1}{27} - \frac{1}{16} + \cdots$$

The kind of behavior illustrated by the series the above example is typical of what happen with any conditionally convergent series. Therefore we must always add the terms of a conditionally convergent series in the order given.

Summary

We have now developed several tests for convergence and divergence of series. In summary:

1. **The nth-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric seies:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating Series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Exercise 15 (Determining Convergence or Divergence).

Which of the following alternating series converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$
2. $\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\sin nx}{n^3}\right]$
3. $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n$
4. $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{n}\right)$
5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$
6. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$
7. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$
8. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$
9. $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}+\cdots$

Exercise 16.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt[n]{10})$
2. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$
3. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$
4. $\sum_{n=1}^{\infty}(-5)^{-n}$
5. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$
6. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$
7. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$
8. $\sum_{n=1}^{\infty}(-1)^{n}\frac{(2n)!}{2^n!n}$
9. $\sum_{n=1}^{\infty}(-1)^{n}(\sqrt{n^2+n}-n)$
10. $\sum_{n=1}^{\infty}(-1)^{n}(\sqrt{n+\sqrt{n}}-\sqrt{n})$
11. $\sum_{n=1}^{\infty}(-1)^{n}\operatorname{sech} n$

Exercise 17.

Determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+3}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(\ln(n+2))}$

Exercise 18.

Use any method to determine whether the series converges or diverges. Give reasons for your answer.

1. $\sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right)$
2. $\sum_{n=2}^{\infty} \frac{3}{10+n^{4/3}}$
3. $\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{n^2}$
4. $\sum_{n=1}^{\infty} \frac{n-2}{n^2+3n} \left(-\frac{2}{3}\right)^n$
5. $\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots$
6. $\sum_{n=3}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$
7. $\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^3$
8. $\sum_{n=2}^{\infty} \frac{1}{1+2+2^2+\cdots+2^n}$
9. $\sum_{n=2}^{\infty} \frac{1+3+3^2+\cdots+3^{n-1}}{1+2+3+\cdots+n}$
10. $\sum_{n=0}^{\infty} (-1)^n \frac{e^n}{e^n+e^{n^2}}$
11. $\sum_{n=1}^{\infty} \frac{4\cdot 6\cdot 8\cdots(2n)}{5^{n+1}(n+2)!}$

Exercise 19.

Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$
3. $\frac{1}{1+t}=\displaystyle\sum_{n=1}^{\infty}(-1)^n t^n, 0<t<1$

Exercise 20.

Approximate the sums in

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n)!}$$

with an error of magnitude less than 5×10^{-6}.

Exercise

Exercise 21.

(a) *The series* $\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \ldots + \frac{1}{3^n} - \frac{1}{2^n} + \ldots$ *Does not meet one of the conditions of Leibniz's Theorem. Which one?*

(b) *Find the sum of the series in part (a).*

Exercise 22.

The limit L of an alternating series that satisfies the conditions of Theorem 3 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n + s_{n+1}}{2} = s_n + \frac{1}{2}(-1)^{n+2} a_{n+1}$$

to estimate L. Compute

$$s_{20} + \frac{1}{2} \cdot \frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.69314718\ldots$*.*

Exercise 23 (The sign of an alternating series that satisfies the conditions of the Leibniz's Theorem).

Prove the assertion in the Alternating Series Estimation Theorem that whenever an alternating series satisfying the conditions of Leibniz's Theorem is approximated with one of its partial sums, then the remainder (sum of then unsaved terms) has the same sign as the first unused term. (Hint: Group the remainder's terms in consecutive pairs.)

Exercise 24.

Show that the sum of the first $2n$ *terms of the series*

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\dots$$

Is the same as the sum of the first n *terms of the series*

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\dots.$$

Do these series converge? What is the sum of the first $2n+1$ *terms of the first series? If the series converge, what is their sum?*

Exercise

Exercise 25.

Show that

$$\sum_{n=1}^{\infty} a_n$$

diverges, then

$$\sum_{n=1}^{\infty} |a_n|$$

diverges.

Exercise 26.

Show that if

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely, then

$$\Big|\sum_{n=1}^{\infty} a_n\Big| \leq \sum_{n=1}^{\infty} |a_n|.$$

Exercise 27.

Show that if $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge absolutely, then so does*

(a) $\sum_{n=1}^{\infty} (a_n + b_n)$

(b) $\sum_{n=1}^{\infty} (a_n - b_n)$

(c) $\sum_{n=1}^{\infty} ka_n$ *(k any number)*

Exercise

Exercise 28.

Show by example that

$$\sum_{n=1}^{\infty} a_n b_n$$

may diverge even if $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge.*

Exercise 29.

1. *If* $\sum a_n$ *converges absolutely, prove that* $\sum a_n^2$ *converges.*
2. *Does the series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$$

   converge or diverge? Justify your answer.

3. *In the alternating harmonic series, suppose the goal is to arrange the terms to get a new series that converges to* $-1/2$*. Start the new arrangement with the first negative term, which is* $-1/2$*. Whenever you have a sum that is less than or equal to* $-1/2$*, start introducing positive terms, taken in order, until the new total is greater than* $-1/2$*. Then add negative terms until the total is less than or equal to* $-1/2$ *again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If* s_n *is the sum of the first n terms of your new series, plot the points* (n, s_n) *to illustrate how the sums are behaving.*

Power Series

Power Series

We have discussed so far some tests for convergence of infinite series. Now we are going to see a special series and see that its sum looks like "infinite polynomials."

We call these sums "power series" because they are defined as infinite series of powers of some variable, in our case x.

Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots . \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Equation (7) is the special case obtained by taking $a = 0$ in Equation (2).

Example 30 (Geometric Series).

Taking all the coefficients to be 1 in Equation (7) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \ldots.$$

This is the geometric series with first term 1 and ratio x.

It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

Up to now, we have use Equation (3) as a formula for the sum of the series on the right.

Power Series and Convergence

We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left.

For values of x near zero, we need take only a few terms of the series to get a good approximation. As we move toward $a = 1$, or -1, we must take more terms.

The following figure show the graphs of $f(x) = 1/(1 - x)$, and the approximating polynomials $y_n = P_n(x)$ for $n = 0, 1, 2$ and 8. The function $f(x) = 1/(1/x)$ is not continuous on intervals containing $x = 1$. where it has a vertical asymptote. The approximations do not apply when $x \geq 1$.

Example 31 (A Geometric Series).

The power series

$$\frac{1}{4} - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \dots \qquad (4)$$

matches equation (2) with

$$a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \cdots, c_n = (-1/2)^n.$$

This is a geometric series with first term 1 *and ratio* $r = -\frac{x-2}{2}$.

The series converges for $|\frac{x-2}{2}| < 1$ or $0 < x < 4$.

The sum is $\frac{1}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{2}{x}$ so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots, \qquad 0 < x < 4.$$

Series (4) generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{x} \\ P_2(x) &= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4} \end{aligned}$$

and so on.

Example 32 (Testing for Convergence Using the Ratio Test).

For what values of x does the following power series converge?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the series in question : $\left|\frac{u_n+1}{u_n}\right| = \frac{n}{n+1}|x| \to |x|$.

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series: it diverges.

The given series converges for $-1 < x \leq 1$ and diverges elsewhere.

Example 33 (Testing for Convergence Using the Ratio Test).

For what values of x does the following power series converge?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the series in question : $\left|\frac{u_n+1}{u_n}\right| = \frac{2n-1}{2n+1}x^2 \to x^2$. The series converges absolutely for $x^2 < 1$. It diverges for $x^2 < 1$ becuase the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Thus the given series converges for $-1 \leq x \leq 1$ and diverges elsewhere.

Example 34 (Testing for Convergence Using the Ratio Test).

For what values of x does the following power series converge?

$$\sum_{n=1}^{\infty} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the series in question : $\left|\frac{u_n+1}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.

Example 35 (Testing for Convergence Using the Ratio Test).

For what values of x does the following power series converge?

$$\sum_{n=1}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the series in question : $\left|\frac{u_n+1}{u_n}\right| = |\frac{(n+1)!x^{n+1}}{n!x^n} = (n+1)|x| \to \infty$ unless $x = 0$.

Theorem 36 (The Convergence Theorem for Power Series).

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

converges for $x = c \neq 0$, *then it converges absolutely for all* x *with* $|x| < |c|$.

If the series diverges for $x = d$. *then it diverges for all* x *with* $|x| > |d|$.

Proof of the Convergence Theorem for Power Series

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$. Hence, there is an integer N such that $|a_n c^n| < 1$ for all $n \geq N$. That is,

$$|a_n| < \frac{1}{|c|^n} \qquad \text{for } n \geq N. \tag{5}$$

Now take any x such that $|x| < |c|$ and consider
$|a_0| + |a_1 x| + \cdots + |a_{N-1} x^{N-1}| + |a_N x^N| + |a_{N+1} x^{N+1}| + \cdots$

There are only a finite number of terms prior to $|a_N x^N|$, and their sum is finite. Starting with $|a_N x^N|$, and beyond, the terms are less than

$$\left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \cdots \tag{6}$$

because of Inequality (5).

Proof of the Convergence Theorem for Power Series (contd...)

But series (6) is a geometric series with ratio $r = |x/c|$, which is less than 1, since $|x| < |c|$. Hence series (6) converges, so the original series converges absolutely. This proves the first half of the theorem.

The second half of the theorem follows from the first. If the diverges at $x = d$ and converges at a value x_0 with $|x_0| > |d|$, we may tacke $c = x_0$ in the first half of the theorem and conclude that the series converges absolutely at d. But the series cannot converge absolutely and diverge at one and the same time. Hence, if it diverges at d, it diverges for all x with $|x| > |d|$.

The Radius of Convergence of a Power Series

The theorem we have just proved and the examples we have studied lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways.

It might converge only at $x = a$, or converge everywhere, or converge on some interval of radius R centered at $x = a$. We prove this as a Corollary to Convergence Theorem for Power Series.

Corollary to Convergence Theorem for Power Series

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Proof of Corollary

We assume first that $a = 0$, so that the power series is centered at 0. If the series converges everywhere we are in Case 2. If it converges only at $x = 0$ we are in Case 3. Otherwise there is a nonzero number d such that $\sum c_n d^n$ diverges. The set S of value of x for which the series $\sum c_n x^n$ converges is nonempty because it contains 0 and a positive number p as well. By Convergence Theorem for Power Series, the sereis diverges for all x with $|x| > |d|$, so $|x| \leq |d|$ for all $x \in S$, and S is a bounded set. By the Completeness Property of the real numbers, a nonempty, bounded set has a least upper bound R. (the least upper bound is the smallest number with the property that the elements $x \in S$ satisty $x \leq R$.) If $|x| > R \geq p$, then $x \neq S$ so the series $\sum c_n x^n$ diverges.

Proof of Corollary (contd...)

If $|x| < R$, then $|x|$ is not an upper bound for S (because it's smaller than the least upper bound) so there is a number $b \in S$ such that $b > |x|$. Since $b \in X$, the series $\sum c_n b^n$ converges and therefore the series $\sum c_n |x|^n$ converges by Convergence Theorem for Power Series. This proves the Corollary for power series centered at $a = 0$.

For a power series centered at $a \neq 0$, we set $x' = (x - a)$ and repeat the argument with $'x'$. Since $x' = 0$ when $x = a$, a radius R interval of convergence for $\sum c_n (x')^n$ centered at $x' = 0$ is the same s a radius R interval of convergence for $\sum c_n (x - a)^n$ centered at $x = a$. This establishes the Corollary for the general case.

Radius of Convergence

R is called **radius of convergence** of the power series and the interval of radius R centered at $x = a$ is called the **interval of convergence**.

The interval of convergence may be open, closed, or half-open, depending on the particular series. At points x with $|x - a| < R$, the series converges absolutely. If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

How to Test a Power Series for Convergence

1. Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval $|x - a| < R$ or $a - R < x < a + R$.
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the Interval of absolute convergence is a $a - R < x < a + R$, the series diverges for $|x - a| > R$ (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

Term-by-Term Differentiation

A Theorem from advanced calculus says that a power series can be differentiated term by term at each interior point of its interval of convergence.

Theorem 37 (The Term-by-Term Differentiation Theorem).

If $\sum c_n(x-a)^n$ converges for a $a-R<x<a+R$ for some $R>0$, it defines a function f : $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, a-R<x<a+R$. Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=0}^{\infty} nc_n(x-a)^{n-1} \qquad f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Example 38 (Applying Term-by-Term Differentiation).

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, -1 < x < 1$$

$$f'(x) \quad = \quad \frac{1}{(1-x^2)} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1},$$

for $-1 < x < 1$

$$f''(x) \quad = \quad \frac{2}{(1-x^3)} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)x^{n-2},$$

for $-1 < x < 1$.

CAUTION

Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n!cos(n!x)}{n^2}$$

which diverges for all x. This is not a power series, since it is not a sum of positive integer powers of x.

Term-by-Term Integration

Another advanced calculate theorem states that a power series can be integrated term by term throughout its interval of convergence.

Theorem 39 (The Term-by-Term Integration Theorem).

Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *converges for* $a - R < x < a + R$ $(R > 0)$*. Then*

$$\sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ *and*

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + c$$

for $a - R < x < a + R$*.*

Example 40 (A series for $\tan^{-1} x, -1 \leq x \leq 1$).

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad -1 \leq x \leq 1.$$

We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \quad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

we can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \quad -1 < x < 1.$$

We will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$.

Notice that the original series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad -1 \leq x \leq 1$ converges at both endpoints of the original interval of convergence, but the "Term-by-Term Integration Theorem" can guarantee the convergence of the differentiated series only inside the interval.

Example 41 (A series for $\ln(1+x), -1 < x \leq 1$).

The series

$$\frac{1}{1+t} = 1 + t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. *Therefore,*

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ *to the number in 2, but that was not guaranteed by the theorem.*

Another theorem from advanced calculus state that absolutely converging power series can be multiplied the way we multiply polynomials. We omit the proof.

Theorem 42 (The series Multiplication Theorem for Power Series).

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$. *and*

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{b-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{\infty} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Proof

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$. Hence, there is an integer N such that $|a_n c^n| < 1$ for all $n \geq N$. That is,

$$|a_n| < \frac{1}{|c|^n} \text{ for } n \geq N. \qquad (5)$$

Now take any x such that $|x| < |c|$ and consider

$$|a_0| + |a_1 x| + \cdots + |a_{N-1} x^{N-1}| + |a_N s^N| + |a_{N+1} x^{N+1}| + \cdots$$

There are only a finite number of terms prior to $|a_N x^N|$, and their sum is finite. Starting with $|a_N x^N|$, and beyond, the terms are less than

$$|\frac{x}{c}|^N + |\frac{x}{c}|^{N+1} + |\frac{x}{c}|^{N+2} + \cdots \qquad (6)$$

because of Inequality (5).

Example 43 (Multiply the Geometric series).

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$, *for* $|x| < 1$, *by itself to get a power series for* $1/(1-x)^2$, *for* $|x| < 1$.

Multiply the Geometric series

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$\begin{aligned} c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0 \\ &= 1 + 1 + \cdots + 1 (n+1 \text{ times}) = n+1 \end{aligned}$$

Multiply the Geometric series

Then, by the Series Multiplication Theorem,

$$\begin{aligned} A(x) \cdot B(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} (n+1) x^n \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots \end{aligned}$$

is the series for $1/(1-x)^2$.

The series all converge absolutely for $|x| < 1$.

$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

Exercise

Exercise 44 (Intervals of Convergence).

In the following exercies, (a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely (c) conditoinally?

1. $\sum_{n=0}^{\infty}(x+5)^n$
2. $\sum_{n=0}^{\infty}\frac{nx^n}{n+2}$
3. $\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$
4. $\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$
5. $\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$

Exercise 45 (Intervals of Convergence).

In the following exercies, (a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely (c) conditoinally?

1. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$
2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$
3. $\sum_{n=1}^{\infty} (\ln n)x^n$
4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$
5. $\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$
6. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$

Exercise 46 (Intervals of Convergence).

In the following exercies, (a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely (c) conditoinally?

1. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$
2. $\sum_{n=0}^{\infty} \frac{(x-\sqrt{2})^{2n+1}}{2^n}$ *In the following exercises, find the series' interval of convergence and, within this interval, the sum of the series as a function of x.*
3. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$
4. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1\right)^n$
5. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

Exercise 47.

For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What it its sum?

Exercise 48.

If you integrate the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

Exercise 49.

The series

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^11}{11!} + \cdots$$

converges to $\sin x$ *for all* x.

(a) *Find the first six terms of a seies for* $\cos x$. *For what values of* x *should the series convege?*

(b) *By replacing* x *by* $2x$ *in the series for* $\sin x$, *find a series that converges to* $\sin 2x$ *for all* x.

(c) *Using the result in part (a) and series multiplication, calculate the first six terms of a series for* $2 \sin x \cos x$. *Compare your answer with the answer in part (b).*

Exercise 50.

The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x *for all* x.

(a) *Find a series for* $(d/dx)e^x$. *Do you get the series for* e^x*?. Explain your answer.*

(b) *Find a series for* $\int e^x dx$. *Do you get the series for* e^x*? Explain your answer.*

(c) *Replace* x *by* $-x$ *in the series for* e^x *to find a series that converges to* e^x *for all* x. *Then multyiply the series for* e^x *and* e^{-x} *to find the six terms of a series for* $e^{-x} \cdot e^x$.

Exercise 51.

The series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

converges to $\tan x$ *for* $-\pi/2 < x < \pi/2$.

(a) *Find the first five terms of the series for* $\ln|\sec x|$. *For what values of* x *should the series converge?*

(b) *Find the first five terms of the series for* $\sec^3$. *Fro what values of* x *should thies series converge?*

(c) *Check your result in part (b) by squaring the series given for* $\sec x$.

Exercise 52.

The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to see x for $-\pi/2 < x < \pi/2$.

(a) *Find the first five terms of a power series for the function in* $|\sec x + \tan x|$. *For what values of x should the series converge?*

(b) *Find the first four terms of a series for* $\sec s \tan x$. *For what values of x should the series converge?*

(c) *Check your result in part (b) by multiplying the series for* $\sec x$ *by the series given for* $\tan x$.

Exercise 53 (Uniqueness of convergent power series).

1. *Show that if two power series* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *are convergent and equal for all values of* x *in an open interval* $(-c, c)$, *then* $a_n = b_n$ *for every* n. *(Hint: Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. *Differentiate term by term to show that* a_n *and* b_n *both equal* $f^{(n)}(0)/(n!)$.*)*
2. *Show that if* $\sum_{n=0}^{\infty} a_n x^n = 0$ *for all* x *in an open interval* $(-c, c)$, *then* $a_n = 0$ *for every* n.

Exercise

Exercise 54 (The sum of the series).

$\sum_{n=0}^{\infty}(n^2/2^n)$ *To find the sum of this series, express* $1/(1-x)$ *as geometric series, differentiate both sides of the resulting equation with respect ot* x*, multiply both sides of the result by* x*, differentiate again, multiply by* x *again, and set* x *equal to* $1/2$*. What do you get? (Source: David E. Dobbs' letter to the editor, Illinois Mathematics Teacher, Vol.33, Issue 4, 1982,p.27.)*

Exercise 55 (Convergence at endpoints).

Show by examples that the convergence of a power series at an endpoint of its interval of convergence, may be either conditional or absolute.

Exercise 56.

Make up a power series whose interval of convergence is

(a) $(-3,3)$

(b) $(-2,0)$

(c) $(1,5)$.

Taylor and Maclaurin Series

Taylor and Maclaurin Series

We now see **how functions that are infinitely differentiable generate power series** called Taylor series.

In many cases, **these series can provide useful polynomial approximations of the generating functions.**

We know that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders.

But what about the other way around?

- If a function $f(x)$ has derivatives of all orders on an interval I, can it be expressed as a power series on I?
- And if it can, what will its coefficients be?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence.

By repeated term-by-term differentiation within the interval of convergence I we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots \\ f''(x) &= 1.2a_2 + 2.3a_3(x-a) + 3.4a_4(x-a)^2 + \cdots \\ f'''(x) &= 1.2.3a_3 + 2.3.4a_4(x-a) + 3.4.5a_5(x-a)^2 + \cdots \end{aligned}$$

With the nth derivate, for all n, being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$,we have

$$\begin{aligned} f'(a) &= a_1 \\ f''(a) &= 1.2a_2 \\ f'''(a) &= 1.2.3a_3. \\ &\vdots \\ f^{(n)}(a) &= n!a_n. \end{aligned}$$

These formulas reveal a pattern in the coefficient of any power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

that converges to the values of f on I ("represents f on I"). If there is such series (still an open question), then there is only one such series and its nth coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a^2) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (7)$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval I centered at $x = a$ and use it to generate the series in Equation (7), will the series then converge to $f(x)$ at each x in the interior of I?

The answer is maybe – for some functions it will but for other functions it will not, as we will see.

Definition 57.

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

The Maclaurin series generated by f is often just called the Taylor series of f.

Example 58.

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

We need to find $f(2), f'(2), f''(2), \cdots$. Taking derivatives we get

$f(x) = x^{-1}$, $\qquad f(x) = 2^{-1} = \frac{1}{2}$,

$f'(x) = -x^{-2}$, $\qquad f'(2) = -\frac{1}{2^2}$,

$f''(x) = 2!x^{-3}$, $\qquad \frac{f''(2)}{3!} = 2^{-3} = \frac{1}{2^3}$,

$f'''(x) = -3!x^{-4}$, $\qquad \frac{f'''(2)}{3!} = -\frac{1}{2^4}$,

$\vdots \qquad \vdots$

$f^{n)}(x) = (-1)^n n! x^{-(n+1)}$, $\qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$.

Finding a Taylor Series

The Taylor series is

$$f(2) + f\prime(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^2 + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots .$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$.

Taylor Polynomials

The linearizion of a differentiable function f at a point a is the polynomial of degree one given by

$$P_1(x) = f(a) + f'(a)(x - a).$$

If f has derivatives of higher order at a, then it has higher-order polynomial approximations as well, one for each available derivative.

These polynomials are called the Taylor polynomials of f.

Taylor Polynomial of order n

Definition 59.

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n *generated by f at $x = a$ is the polynomial*

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Polynomial of order n

We speak of a Taylor polynomial of order n rather than degree n because $f^{(n)}(a)$ may be zero.

The first two Taylor polynomials of $f(x) = \cos x$ at $x = 0$, for example, are $P_0(x) = 1$ and $P_1(x) = 1$.

The first-order Taylor polynomial has degree zero, not one.

Just as the linearization of f at $x = a$ provides the best linear approximation of f in the neighborhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of thier respective degrees.

Example 60 (Finding Taylor Polynomials for e^x).

Find the Taylor series the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution : Since $f(x) = e^x, \quad f'(x) = e^x, \quad \ldots, \quad f^{(n)}(x) = e^x, \ldots,$ we have $f(0) = e^0 = 1, \quad f'(0) = 1, \quad \ldots, \quad f^{(n)}(0) = 1, \quad \ldots.$

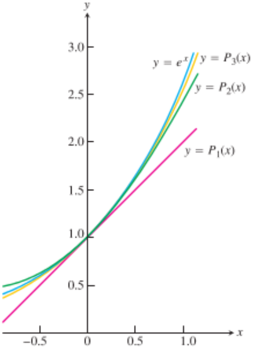
The Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This is also the Maclaurin series for e^x. **We will later see that the series converges to e^x at every x.**

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$



The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Finding Taylor Polynomials for $\cos x$

Example 61.

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ *at* $x = 0$.

Solution : The cosine and its derivatives are

$$f(x) = \cos x, \qquad f'(x) = -\sin x,$$

$$f''(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$

$$f^{(2n)}(x) = (-1)^n \cos x, \qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x.$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

Finding Taylor Polynomials for $\cos x$

The Taylor series generated by f at 0 is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 0.x - \frac{x^2}{2!} + 0.x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

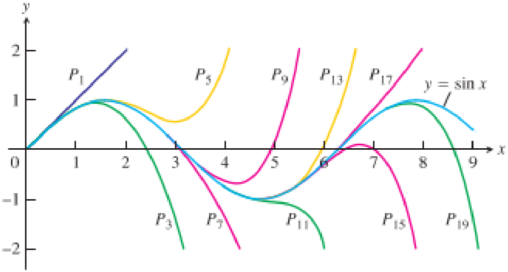
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)}!$$

This is also the Maclaurin series for cos x. **We will see that the series converges to $\cos x$ at every x.**

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n+1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

The following figure shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis.



The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \to \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \leq 1$,

A Function f Whose Taylor Series Converges at Every x but Converges to $f(x)$ Only at $x = 0$

The following example shows that there is a function f whose Taylor series converges at every x but converges to $f(x)$ only at $x = 0$.

Example 62.

It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

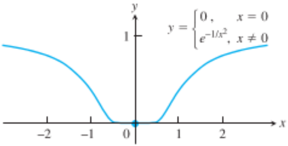
has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n.

A function f whose Taylor series converges at every x but converges to $f(x)$ only at $x = 0$.

This means that the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = 0 + 0.x + 0.x^2 + \cdots + 0.x^n + \cdots = 0.$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$.



The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore its Taylor series is not the function itself.

Some questions

Two questions still remain

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor, which will be discussed next.

Exercise 63.

In the following exercises, find the Taylor polynomials of orders $0, 1, 2$, *and* 3 *generated by* f *at* a.

1. $f(x) = \ln x.$ $\quad a = 1$
2. $f(x) = 1/(x+2),$ $\quad a = 0$
3. $f(x) = \sin x,$ $\quad a = \pi/4$
4. $f(x) = \sqrt{x},$ $\quad a = 4$
5. $f(x) = \sqrt{x+4},$ $\quad a = 0$

Finding Taylor Series at x=0 (Maclaurin Series)

Exercise 64.

Find the Maclaurin series for the function in the following exercises.

1. e^{-x}
2. $\frac{1}{1-x}$
3. $\sin \frac{x}{2}$
4. $7\cos(-x)$
5. $\sinh x = \frac{e^x - e^{-x}}{2}$
6. $x^4 - 2x^3 - 5x + 4$

Exercise 65.

In the following exercises, find the Taylor series generated by f at $x = 0$.

1. $f(x) = x^3 - 2x + 4, \qquad a = 2$
2. $f(x) = x^4 + x^2 + 1, \qquad a = -2$
3. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \qquad a = -1$
4. $f(x) = x/(1 - x), \qquad a = 0$
5. $f(x) = e^x, \qquad a = 2$
6. $f(x) = 2^x, \qquad a = 1$

Exercise 66.

(a) *Use the Taylor series generated by* e^x *at* $x = a$ *to show that*

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots\right].$$

(b) *Find the Taylor series generated by* e^x *at* $x = 1$. *Compare your answer with the formula in the above exercise.*

(c) *Let* $f(x)$ *have derivatives through order* n *at* $x = a$. *Show that the Taylor polynomial of order* n *and its first* n *derivatives have the same values that* f *and its first* n *derivatives have at* $x = a$.

Of all polynomials of degree $\leq$ n, the Taylor polynomial of order n gives the best approximation

Exercise 67.

Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$ is a polynomial of degree n with constant coefficient $b_0, \ldots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g conditions

(a) $E(a) = 0$ *(the approximation error is zero at $x = a$)*

(b) $\lim_{x \to a} \frac{E(x)}{(x-a)^n} = 0$ *(the error is negligible when compare to $(x-a)^n$)*

then

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x-a)^n$.

Exercise 68.

The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the quadratic approximation *of f at $x = a$. Find the*

(a) *linearizion (Taylor polynomial of order 1) and*

(b) *quadratic approximation of f at $x = 0$.*

in the following exercises.

1. $f(x) = \ln(\cos x)$
2. $f(x) = e^{\sin x}$
3. $f(x) = 1/\sqrt{1 - x^2}$
4. $f(x) = \cosh x$
5. $f(x) = \sin x$
6. $f(x) = \tan x$

Convergence of Taylor Series

Convergence of Taylor Series

We now address the following two questions.

1. When does a Taylor series converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

Taylor's Theorem

We answer the questions with the following theorem.

Theorem 69.

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem.

Proof of the theorem will be discussed at the end of this lecture.

Taylor's Formula

When we apply Taylor's Theorem, we usually want to hold "a" fixed and treat "b" as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x. Here is a version of the theorem with this change.

Theorem 70.

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{8}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{9}$$

for some c between a and x.

Taylor's Formula

When we state Taylor's theorem this way, it says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is determined by the value of the $(n+1)$st derivative $f^{(n+1)}$ at a point c that depends on both a and x, and that lies somewhere between them.

For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I.

Taylor's Formula

Equation (8) is called **Taylor's formula**. The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I.

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Often we can estimate R_n without knowing the value of c, as the following example illustrates.

Example 71.

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution. The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (8) and (9) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

and

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between 0 and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$.

Example

Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0, \qquad e^c < 1$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0. \quad e^c < e^x$$

Finally, because

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x,$$

$\lim\limits_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots . \tag{10}$$

Example

We can use the result of Example 71 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

where for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!} \qquad \textit{since} \quad e^c < e^1 < 3.$$

Estimating the Remainder

It is often possible to estimate $R_n(x)$ as we did in Example 71. This method of estimation is so convenient that we state it as a theorem for future reference.

Theorem 72 (The Remainder Estimation Theorem).

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Exercise

We are now ready to look at some examples of how the Remainder Estimation Theorem and Taylor's Theorem can be used together to settle questions of convergence. As we will see, they can also be used to determine the accuracy with which a function is approximated by one of its Taylor polynomials.

Example 73.

Show that the Taylor series for $\sin x$ *at* $x = 0$ *converges for all* x*.*

Solution. The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

Solution (contd...)

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $(|x|^{2k+2}/(2k+2)!) \to 0$ as $k \to \infty$, whatever the value of x, so $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x. Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Example 74.

Show that the Taylor series for $\cos x$ *at* $x = 0$ *converges to* $\cos x$ *for every value of* x*.*

Solution. We add the remainder term to the Taylor polynomial for $\cos x$ to obtain Taylor's formula for $\cos x$ with $n = 2k$;

$$\cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \cdots + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1.\frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k}(x) \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos\ x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots . \qquad (11)$$

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 75.

Using known series, find the first few terms of the Taylor series for the given function using power series operations.

(a) $\frac{1}{3}(2x + x \cos x)$

(b) $e^x \cos x$

Solution

$$(a) \quad \frac{1}{3}(2x + x \ \cos \ x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots\right)$$
$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3.4!} - \cdots - x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$$

$$(b) \quad e^x \cos \ x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right).\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$
$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} \cdots\right)$$
$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots\right) + \cdots$$
$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

Using Taylor series

We recall that if $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then

$$\sum_{n=0}^{\infty} a_n (f(x))^n$$

converges absolutely for any continuous function f on $|f(x)| < R$.

We can use the Taylor series of the function f to find the Taylor series of $f(u(x))$ where $u(x)$ is any continuous function.

The Taylor series resulting from this substitution will converge for all x such that $u(x)$ lies within the interval of convergence of the Taylor series of f.

Using Taylor series

For instance, we can find the Taylor series for $\cos\ 2x$ by substituting $2x$ for x in the Taylor series for $\cos\ x$:

$$\cos\ 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}.$$

Example 76.

For what values of x can we replace $\sin x$ by $x - \left(x^3/3!\right)$ with an error of magnitude no greater than 3×10^{-4}?

Solution. Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem, the error in truncating

$$\sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

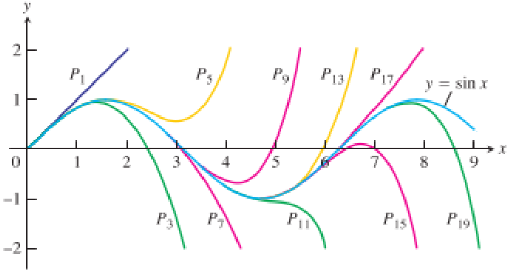
after $\left(x^3/3!\right)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{306 \times 10^{-4}} \approx 0.514.$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - \left(x^3/3!\right)$ for $\sin x$ is an underestimate when x is positive, because then $x^5/120$ is positive.



The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \to \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \leq 1$,

The above figure shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials.

The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $0 \leq x \leq 1$.

Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$. The proof for $a > b$ is nearly the same. The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and its first n derivatives match the function f and its first n derivatives at $x = a$. We do not disturb that matching if we add another term of the form $K(x-a)^{n+1}$, where K is any constant, because such a term and its first n derivatives are all equal to zero at $x = a$. The new function

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}$$

and its first n derivatives still agree with f and its first n derivatives at $x = a$.

Proof of Taylor's Theorem (contd...)

We now choose the particular value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b-a)^{n+1}, \quad or \quad K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}. \tag{12}$$

With K defined by Equation (12), the function

$$F(x) = f(x) - \phi_n(x)$$

measures the difference between the original function f and the approximating function ϕ_n for each x in $[a, b]$.

Proof of Taylor's Theorem (contd...)

We now use Rolle's Theorem. First, because $F(a) = F(b) = 0$ and both F and F' are continuous on $[a, b]$, we know that

$$F'(c_1) = 0$$

for some c_1 in (a, b). Next, because $F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$ we know that

$$F''(c_2) = 0$$

for some c_2 in (a, c_1). Rolle's Theorem, applied successively to $F'', F''', \ldots, F^{(n-1)}$ implies the existence of

$$\begin{array}{llll} c_3 & \text{in } (a, c_2) & \text{such that} & F'''(c_3) = 0, \\ c_4 & \text{in } (a, c_3) & \text{such that} & F^{(4)}(c_4) = 0, \\ \vdots & & & \\ c_n & \text{in } (a, c_{n-1}) & \text{such that} & F^{(n)}(c_n) = 0. \end{array}$$

Proof of Taylor's Theorem (contd...)

Finally, because $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n), and $F^{(n)}(a) = F^{(n)}(c_n) = 0$, Rolle's Theorem implies that there is a number c_{n+1} in (a, c_n) such that

$$F^{(n+1)}(c_{n+1}) = 0 \tag{13}$$

If we differentiate $F(x) = f(x) - P_n(x) - K(x-a)^{n+1}$ a total of $n+1$ times, we get

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!K. \tag{14}$$

Proof of Taylor's Theorem (contd...)

Equations (13) and (14) together give

$$K = \frac{f^{(n+1)}(c)}{(n+1)!} \quad \text{for some number} \quad c = c_{n+1} \text{ in } (a, b)\,. \tag{15}$$

Equation (12) and (15) give

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\,.$$

This concludes the proof.

Exercises 77.

Use substitution to find the Taylor series at $x = 0$ of the functions in the following exercises.

1. $\sin\left(\frac{\pi x}{2}\right)$
2. $\cos\left(x^{2/3}/\sqrt{2}\right)$
3. $\tan^{-1}(3x^4)$
4. $\frac{1}{1+\frac{3}{4}x^3}$

Exercises 78.

Use power series operations to find the Taylor series at $x = 0$ for the functions in the following exercises.

1. xe^x
2. $\frac{x^2}{2} - 1 + \cos x$
3. $x \ln(1 + 2x)$
4. $\sin x \cdot \cos x$
5. $\cos x - \sin x$
6. $\ln(1 + x) - \ln(1 - x)$

Exercises 79.

Find the first four nonzero terms in the Maclaurin series for the functions in the following exercises.

1. $\frac{\ln(1+x)}{1-x}$
2. $(\tan^{-1} x)^2$
3. $\cos^2 x \cdot \sin x$
4. $\sin(\tan^{-1} x)$

Exercises 80.

1. *Estimate the error if* $P_3(x) = x - (x^3/6)$ *is used to estimate the value of* $\sin x$ *at* $x = 0.1$.
2. *Estimate the error if* $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ *is used to estimate the value of* e^x *at* $x = 1/2$.
3. *For approximately what values of* x *can you replace* $\sin x$ *by* $x - (x^3/6)$ *with an error of magnitude no greater than* 5×10^{-4}*? Give reasons for your answer.*
4. *If* $\cos x$ *is replaced by* $1 - (x^2/2)$ *and* $|x| < 0.5$, *what estimate can be made of the error? Does* $1 - (x^2/2)$ *tend to be too large, or too small? Give reasons for your answer.*

Exercises 81.

1. *How close is the approximation* $\sin x = x$ *when* $|x| < 10^{-3}$*? For which of these values of* x *is* $x < \sin x$*?*
2. *The estimate* $\sqrt{1+x} = 1 + (x/2)$ *is used when* x *is small. Estimate the error when* $|x| < 0.01$.
3. *The approximation* $e^x = 1 + x + (x^2/2)$ *is used when* x *is small. Use the Remainder Estimation Theorem to estimate the error when* $|x| < 0.1$.
4. *(Continuation of the above exercise) When* $x < 0$*, the series for* e^x *is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing* e^x *by* $1 + x + (x^2/2)$ *when* $-0.1 < x < 0$*. Compare your estimate with the one you obtained in the above exercise.*

Exercises 82.

1. *Use the identity* $\sin^2 x = (1 - \cos 2x)/2$ *to obtain the Maclaurin series for* $\sin^2 x$. *Then differentiate this series to obtain the Maclaurin series for* $2 \sin x \cos x$. *Check that this is the series for* $\sin 2x$.
2. *(Continuation of the above xercise.) Use the identity* $\cos^2 x = \cos 2x + \sin^2 x$ *to obtain a power series for* $\cos^2 x$.
3. Taylor's Theorem and the Mean Value Theorem. *Explain how the Mean Value Theorem is a special case of Taylor's Theorem.*
4. Linearizations at inflection points. *Show that if the graph of a twice-differentiable function* $f(x)$ *has an inflection point at* $x = a$, *then the linearization of* f *at* $x = a$ *is also the quadratic approximation of* f *at* $x = a$. *This explains why tangent lines fit so well at inflection points.*

Exercises 83.

1. The (second) second derivative test. *Use the equation*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2$$

to establish the following test:
Let f have continuous first and second derivatives and suppose that $f'(a) = 0$*. Then*

   a. *f has a local maximum at a if* $f'' \leq 0$ *throughout an interval whose interior contains a;*
   b. *f has a local maximum at a if* $f'' \geq 0$ *throughout an interval whose interior contains a.*

Exercises 84.

1. A cubic approximation. *Use Taylor's formula with* $a = 0$ *and* $n = 3$ *to find the standard cubic approximation of* $f(x) = 1/(1 - x)$ *at* $x = 0$. *Give an upper bound for the magnitude of the error in the approximation when* $|x| \leq 0.1$.
2. a. *Use Taylor's formula with* $n = 2$ *to find the quadratic approximation of* $f(x) = (1 + x)^k$ *at* $x = 0$ *(k a constant).*
   b. *If* $k = 3$, *for approximately what values of* x *in the interval* $[0, 1]$ *will the error in the quadratic approximation be less than* $1/100$*?*
3. Improving approximations of π:
   a. *Let* P *be an approximation of* π *accurate to* n *decimals. Show that* $P + \sin P$ *gives an approximation correct to* $3n$ *decimals. (Hint : Let* $P = \pi + x$.*)*
   b. *Try it with a calculator.*

Exercise 85.

A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ has a Taylor series that converges to the function at every point of $(-R, R)$. Show this by showing that the Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $\sum_{n=0}^{\infty} a_n x^n$ itself.
An immediate consequence of this is that series like

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots$$

and

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \dots,$$

obtained by multiplying Taylor series by powers of x, as well as series obtained by integration and differentiation of convergent power series, are themselves the Taylor series generated by the functions they represent.

Exercises 86.

1. Taylor series for even functions and odd functions. *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for all* x *in an open interval* $(-R, R)$. *Show that*
   a. *If* f *is even, then* $a_1 = a_3 = a_5 = \cdots = 0$*, i.e., the Taylor series for* f *at* $x = 0$ *contains only even powers of* x.
   b. *If* f *is odd, then* $a_0 = a_2 = a_4 = \cdots = 0$*, i.e., the Taylor series for* f *at* $x = 0$ *contains only odd powers of* x.

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